CHAPTER 3

How to Count

1. One, two, three...

Youtube.com has a video at this URL

`http://www.youtube.com/watch?v=wcCw9RHI5mc`

The important part is that "`v=wcCw9RHI5mc`" business at the end, which essentially means "this is video number `wcCw9RHI5mc`". This video is, of course, different than number `wcCw9RHI5m`*d*, and number `wcCw9RHI5m`*e* and so on. We can notice that the video number contains 11 different slots (count them), each of which is filled with a number or upper or lower case Latin letter, which means the number is case sensitive; `A` differs from `a`. The question is, how many different videos can Youtube host given this numbering scheme? Are they going to run out of numbers anytime soon?

That problem is hard, so we'll start on a simpler one. Suppose the video numbering scheme only allowed one slot, and that this slot could only contain a single-digit number, chosen from 0-9. Then how many videos could they host? They'd have `v=0`, `v=1` and so on. Ten, right? Now how about if they allowed two slots chosen from 0-9. Just 10 for the first, and 10 for each of the 10 of the first, a confusing way of saying 10×10. For three slots it's $10 \times 10 \times 10$. But you already knew how to do this kind of counting, didn't you?

Suppose the single slot is allowed only to be the lower case letters `a,...,z`? This is `v=a`, `v=b`, etc. How many in two such slots? Just $26 \times 26 = 676$. Which is the same way we got 100 in two slots of the numbers 0-9.

So if allow any number, plus any lower or upper case letter in any slot, we have $10+26+26=62$ different possibilities per slot. That means that with 11 slots we have $62 \times 62 \cdots \times 62 = 62^{11} \approx 5 \times 10^{19}$, or 50 billion billion different videos that Youtube can host.

2. Arrangements

How many ways are there of arranging things? In 1977, George Thorogood remade that classic John Lee Hooker song, "One Bourbon, One Scotch, and One Beer." This is because George is, of course, the spiritous counterpart of an oenophile; that is, he is a connoisseur of fine spirits and regularly participates in tastings. Further, George, who is way past 21, is not an idiot and never binge drinks, which is about the most moronic of activities that a person could engage in. He very much wants to arrange his coming week, where he will taste, each night, one bourbon (B) , one scotch (S), and one beer (R). But he wants to be sure that the order he tastes these drinks doesn't influence his personal ratings. So each night he will sip them in a different order. How many different nights will this take him? Write out what will happen: Night 1, BSR; night 2, BRS; night 3, SBR; night 4, SRB; night 5, RBS; night 6, RSB. Six nights! Luckily, this still leaves Sunday free for contemplation.

Later, George decides to broaden his tasting horizons by adding Vernors (the tasty ginger ale aged in oak barrels that can't be bought in New York City) to his line up. How many nights does it take him to taste things in different order now? We could count by listing each combination, but there's an easier way. If you have n items and you want to know how many different ways they could be grouped or ordered, the general formula is:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \tag{9}$$

The term on the left, $n!$, reads "n factorial." With 4 beverages, this is $4 \times 3 \times 2 \times 1 = 24$ nights, which is over three weeks! Good thing that George is dedicated.

3. Being choosy

It's the day before Thanksgiving and you are at school, packing your car for the drive home. You would have left a day earlier, but you didn't want to miss your favorite class—statistics. It turns out that you have three friends who you know need a ride: Larry, Curly, and Moe. Lately, they have been acting like a bunch of stooges, so you decide to tell them that your car is just too full to bring them along. The question is, how many different ways can you arrange your friends to drive home with you when you plan to bring none of them? This is not a trick question; the answer is as easy as you think. Only one way—that is, with you driving alone.

But, they are your friends, and you love them, so you decide to take just one. Now how many ways can you arrange your friends so that you take just one? Since you can take Larry, Curly, or Moe, and only one, then it's obviously three different ways, just by taking only Larry, or only Curly, or only Moe. What if you decide to take two, then how many ways? That's trickier. You might be tempted to think that, given there are 3 of them, that the answer is $3! = 6$, but that's not quite right. Write out a list of the groupings: you can take Larry & Curly, Larry & Moe, or Moe & Curly. That's three possibilities. The grouping "Curly & Larry," for example, is just the same as the grouping "Larry & Curly." That is, the order of your friends doesn't matter: this is why the answer is 3 instead of 6. Finally, all these calculations have made you so happy that you soften your heart and decide to take all three. How many different groupings taking all of them are possible? Right. Only one.

You won't be surprised to learn that there is a formula to cover situations like this. If you have n friends and you want to count the number of possible groupings of k of them when the order does not matter, then the formula is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \tag{10}$$

The term on the left is read "n choose k". By definition (via some fascinating mathematics) $0! = 1$. Here are all the answers for the

Thanksgiving problem:

$$\binom{3}{0} = \frac{3!}{3!0!} = 1 \qquad \binom{3}{1} = \frac{3!}{2!1!} = 3$$
$$\binom{3}{2} = \frac{3!}{1!2!} = 3 \qquad \binom{3}{3} = \frac{3!}{1!3!} = 1$$

There are some helpful facts about this combinatorial function that are useful to know. The first is that $\binom{n}{0}$ always equals 1. This means, out of n things, you take none; or it means there is only one way to arrange no things, namely no arrangement at all. $\binom{n}{n}$ is also always 1, regardless of what n equals. It means, out of n things, you take all. $\binom{n}{1}$ always equals n, and so does $\binom{n}{n-1}$: these are the number of ways of choosing just 1 or just $n-1$ things. As long as $n > 2$, $\binom{n}{2} > \binom{n}{1}$, which makes sense, because you can make more groups of 2 than of 1.

4. Counting meets probability: The Binomial distribution

We started the Thanksgiving problem by considering it from your point of view. Now we take Larry, Moe, and Curly's perspective, who are waiting in their dorm room for your call. They don't yet know whether which, or if any of them, will get a ride with you. Because they do not know, they want to quantify their uncertainty and they do so using probability. We are now entering a different realm, where counting meets probability. Take your time here, because the steps we follow will the same in *every* probability problem we ever do.

Moe, reminiscent, recalls an incident wherein he was obliged to poke you in the eyes, and guesses that, since you were somewhat irked at the time, the probability that you take any one of the gang along is only 10%. That is, it is his judgment that the probability that you take him, Moe, is 10%, which is the same as you would also (independently) take Curly and so on. So the boys want to figure out the probability that you take none of them, take one of them, take two of them, or take all three of them.

Start with taking all three. We want the probability that you take Larry *and* Moe *and* Curly, where the probability of taking each is 10%. Remember probability rule #2? Those "ands" become "times": so the probability of taking all three is $0.1 \times 0.1 \times 0.1 = 0.001$, or 1 in a 1000. Keep in mind: this is from *their* perspective, not yours. This is their guess of the chances; because you may already have made up your mind—but they don't know that.

What about taking none of them? This is the chance that you do not take Larry *and* you do not take Moe, *and* you do not take Curly. The key word is still "and;" which makes the probability $(1-0.1)\times(1-0.1)\times(1-0.1) = 0.9^3 \approx 0.73$, since the probability of not taking Larry etc. is one minus the probability of taking him etc. It is, too, because you can either take Larry or not; these are the only two things that can happen, so the probability of taking Larry or not *must* be 1. We can write this using our notation: let A = "Take Larry", then A^F = "Don't take him". Then $\Pr(\mathrm{A}\cup\mathrm{A}^F|\mathrm{E}) = \Pr(\mathrm{A}|\mathrm{E}) + \Pr(\mathrm{A}^F|\mathrm{E}) = 1$, using probability rule #1. So if $\Pr(\mathrm{A}|\mathrm{E}) = 0.1$, then $\Pr(\mathrm{A}^F|\mathrm{E}) = 1-\Pr(\mathrm{A}|\mathrm{E}) = 0.9$. In this case, E is the information dictated by Moe (who is the leader), which lead him to say $\Pr(\mathrm{A}|\mathrm{E}) = 0.1$.

How about taking just one? Well, you can take Larry, not take Moe, and not take Curly, and the chance of that is (using rules #1 and #2 together) $0.1\times(1-0.1)\times(1-0.1)\approx 0.08$; but you could just as easily have taken Moe and not Larry, or Curly and not Larry, and the chance you do either of these is just the same as you taking Larry and not the other two. For shorthand, write M as "Take M" and so on, and M^F as not take M and so on. Thus you could "$\mathrm{LM}^F\mathrm{C}^F$ *or* $\mathrm{L}^F\mathrm{MC}^F$ *or* $\mathrm{L}^F\mathrm{M}^F\mathrm{C}$." Using probability rule #1, we break up this statement into three pieces ("$\mathrm{LM}^F\mathrm{C}^F$"), and then use probability rule #2 on each piece ("ands" turn to times), then add the whole thing up.

You could do all that, but there is an easier way. You could notice there are three different ways to take just one—which we remember from our choosing formula, eq. (10). This makes the probability $\binom{3}{1}0.08 = 3\times 0.08 = 0.24$. Since we already know the probability of taking one of those combinations, we just multiply it by the number of times we see it. We could have also written the answer like this:

$$\binom{3}{1}0.1^1(1-0.1)^2 = 0.24.$$

And we could also written the first situation (taking all of them) in the same way

$$\binom{3}{0}0.1^3(1-0.1)^0 = 0.001.$$

where you must remember that $a^0 = 1$ (for any a you will come across).

You see the pattern by now. This means we have another formula to add to our collection. This one is called the binomial and it looks like this:

$$\Pr(k|n,p,\mathrm{E}_B) = \binom{n}{k}p^k(1-p)^{n-k}. \tag{11}$$

There is a subtle shift in notation with this formula, made to conform with tradition. "k" is shorthand for the statement, in this instance, K = "You take k people." For general situations, k is the number of "successes": or, K = "The number of successes is k". Everything to the right of the "|" is still information that we know. So n is shorthand for N = "There are n possibilities for success", or in your case, N = "There are three brothers which could be taken." The p means, P = "The probability of success is p". We already know E_B, written here with a subscript to remind us we are in a binomial situation. This new notation can be damn convenient because, naturally, most of the time statisticians are working with numbers, and the small letters mean "substitute a number here," and if statisticians are infamous for their lack of personality, at least we have plenty of numbers. This notation can cause grief, too. Just how that is so must wait until later.

Don't forget this: in order for us to be able to use a binomial distribution to describe our uncertainty, we need three things. (1) The definition of a *success*: in the Thanksgiving example, a success was a person getting a ride. (2) The probability of a success is always the same. (3) The number of chances for successes is fixed.

5. Homework

(1) It turns out that Youtube actually allows more than just numbers and letters in their video numbers. They also use the symbol "_" (the underscore). Now how many videos can they host?

(2) In the 23 April 2008 *Wall Street Journal*, on the front page, Ford Motor Company CEO Alan Mulally complained about the "mind-boggling level of vehicle customization, which jacked up costs. Until recently, for instance, the Lincoln Navigator offered 128 options on its console alone." How many differently optioned Lincoln Navigators can be built?

(3) The daily lottery in New York requires you to pick three different numbers to win. How many different combinations of three numbers are there? What are the chances you win?

(4) You just got a new dorm room, and have three roommates, and two bunkbeds. How many different sleeping arrangements are there, assuming, I hope it isn't necessary to say, one per bed? Later, one of your roommates (a football linesman who rarely bathes) insists, in an emphatic way, that he *must* have the top bunk facing East. How many arrangements now?

(5) The FAA uses three-capital-letter designators for airport codes; for example, LGA is La Guardia and DTW is Detroit Metro. How many unique airport codes can there be?

(6) You are staying away from home at college for the first time, and have decided to re-invent yourself. Nobody here knows that you were the kid that mistakenly ate part of your classmate's science project. Time to start fresh. So you buy an entire new wardrobe, consisting of six shirts, and three pants. Assuming you'll have to wear one shirt and one pair of pants to create an ensemble, how many different ensembles can you wear?

(7) You have discovered that pair of pants 1 does *not* go with shirts 3 and 4; and that pants 2 does not go with shirts 1, 5, or 6. How many ensembles are now possible?

(8) You are a generous soul and decide to forgive Moe and decide to take all the gang with you. How many seating arrangements are there, assuming you drive the car?

(9) Part of the binomial formula is $p^k(1-p)^{n-k}$. Explain how this part comes about, that is, why is it this and not something else? I mean, why are the exponents on p and $(1-p)$ k and

$n-k$. Can you explain it in terms of the probability rules you already know?

(10) Another lottery question. The multiple-state Mega Millions drawing requires you to guess 5 different numbers from 1 to 56. It also requires you to pick a "mega" number (after those 5) from 1 to 46. In the first case, what is the chance that you guess the first number correctly?

(11) EXTRA: Greek sororities and fraternities are designated by two or three Greek letters, like $\Gamma\Omega$ or $\Sigma\Pi\Delta$. How may Greek unique societies are possible?

(12) EXTRA: Suppose Moe estimates the probability that you take him as 0.1, and Larry too. But since Curly knows he's so lovable, he estimates his probability of going at 0.8. What is the probability (they estimate) that you take none, one, two, or all three. HINT: Do *not* use the binomial.

(13) EXTRA: You can see that $\binom{n}{2} > \binom{n}{1}$. With some playing around, it's easy to see that $\binom{n}{3} > \binom{n}{2}$. But it's also true that $\binom{n}{n-1} > \binom{n}{n}$ and $\binom{n}{n-2} > \binom{n}{n-1}$. Can you find an m such that $\binom{n}{m}$ is larger than any other $\binom{n}{k}$ where $k \neq m$? If you cannot find it, at least make a guess and give a reason why you chose that guess.